

Lower bounds on Information Divergence

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Abstract—In this paper we establish lower bounds on information divergence from a distribution to certain important classes of distributions as Gaussian, exponential, Gamma, Poisson, geometric, and binomial. These lower bounds are tight and for several convergence theorems where a rate of convergence can be computed, this rate is determined by the lower bounds proved in this paper. General techniques for getting lower bounds in terms of moments are developed.

I. INTRODUCTION AND NOTATIONS

In 2004, O. Johnson and A. Barron have proved [1] that the rate of convergence in the information theoretic Central Limit Theorem is upper bounded by c/n under suitable conditions. P. Harremoës extended this work in [2] based on a maximum entropy approach. Similar results have been obtained for the convergence of binomial distributions to Poisson distributions. Finally the rate of convergence of convolutions of distributions on the unit circle toward the uniform distribution can be bounded. In each of these cases lower bounds on information divergence in terms of moments of orthogonal polynomials or trigonometric functions give lower bounds on the rate of convergence. In this paper, we provide more lower bounds on information divergence using mainly orthogonal polynomials and the related exponential families.

We will identify $x!$ with $\Gamma(x+1)$ even when x is not an integer. Similarly the generalized binomial coefficient $\binom{x}{n}$ equals $x(x-1)\cdots(x-n+1)/n!$ when x is not an integer. We use τ as short for 2π .

II. MOMENT CALCULATIONS

Let $\{Q_\beta; \beta \in \Gamma\}$ denote an exponential family of distributions such that the Radon-Nikodym derivative is

$$\frac{dQ_\beta}{dQ_0} = \frac{\exp(\beta \cdot x)}{Z(\beta)}$$

and where Γ is the set of β such that the partition function Z is finite, i.e.

$$Z(\beta) = \int \exp(\beta \cdot x) dQ_0(x) < \infty.$$

The partition function Z is also called the *moment generating function*. The parametrization $\beta \rightarrow Q_\beta$ is called the natural parametrization. The mean value of the distribution Q_β will be denoted μ_β . The distribution with mean value μ is denoted Q^μ so that $Q^{\mu_\beta} = Q_\beta$. The inverse of the function $\beta \rightarrow \mu_\beta$ is denoted $\hat{\beta}(\cdot)$ and equals the maximum likelihood estimate of the canonical parameter. The variance of x with respect to Q^μ is denoted $V(\mu)$ so that $\mu \rightarrow V(\mu)$ is the *variance function*

of the exponential family. This variance function uniquely characterizes the exponential family.

We note that $\beta \rightarrow \ln Z(\beta)$ is the *cumulant generating function* so that

$$\begin{aligned} \frac{d}{d\beta} \ln Z(\beta)|_{\beta=0} &= E[X], \\ \frac{d^2}{d\beta^2} \ln Z(\beta)|_{\beta=0} &= \text{Var}(X), \\ \frac{d^3}{d\beta^3} \ln Z(\beta)|_{\beta=0} &= E[(X - E[X])^3]. \end{aligned}$$

Lemma 1. Let $\{Q_\beta; \beta \in \Gamma\}$ denote an exponential family with

$$\frac{dQ_\beta}{dQ_0} = \frac{\exp(\beta \cdot x)}{Z(\beta)}.$$

Then

- 1) for all μ and ν ,

$$D(Q^\mu \| Q^\nu) = \frac{(\mu - \nu)^2}{2V(\eta)}$$

for some η between μ and ν .

- 2) for all α and $\beta \in \Gamma$,

$$D(Q_\alpha \| Q_\beta) = \frac{V(\mu_\gamma)}{2} (\alpha - \beta)^2$$

for some γ between α and β , and

Proof: The two parts of the theorem are proved separately.

- 1) We consider the function

$$\begin{aligned} g(t) &= D(Q^t \| Q^\nu) \\ &= \left(\hat{\beta}(t) - \hat{\beta}(\nu) \right) \cdot t \\ &\quad + \ln Z(\hat{\beta}(\nu)) - \ln Z(\hat{\beta}(t)). \end{aligned}$$

The two first derivatives of this function are

$$\begin{aligned} g'(t) &= \frac{d\hat{\beta}(t)}{dt} t + \left(\hat{\beta}(t) - \hat{\beta}(\nu) \right) \\ &\quad - \frac{Z'(\hat{\beta}(t))}{Z(\hat{\beta}(t))} \frac{d\hat{\beta}(t)}{dt} \\ &= \hat{\beta}(t) - \hat{\beta}(\nu), \\ g''(t) &= \frac{1}{dt/d\hat{\beta}(t)} = \frac{1}{V(t)}. \end{aligned}$$

According to Taylor's formula there exists η between μ and ν such that

$$\begin{aligned} D(Q^\mu \| Q^\nu) &= g(\nu) + (\mu - \nu) f'(\nu) + \frac{1}{2} (\mu - \nu)^2 f''(\eta) \\ &= \frac{(\mu - \nu)^2}{2V(\eta)}. \end{aligned}$$

2) The second part is proved in the same way as the first part. \blacksquare

Corollary 2. Let $\beta \rightarrow Q_\beta, \beta \in \Gamma$ denote an exponential family with

$$\frac{dQ_\beta}{dQ_0} = \frac{\exp(\beta \cdot x)}{Z(\beta)}.$$

If the variance function of the exponential family is increasing then

$$D(Q^\mu \| Q^\nu) \geq \frac{(\mu - \nu)^2}{2V(\nu)}$$

for $\mu \leq \nu$.

The binomial distributions, Poisson distributions, geometric distributions, negative binomial distributions, inverse binomial distributions, and generalized Poisson distributions are exponential families with at most cubic variance functions [3], [4]. Using the former corollary we can provide a lower bound on information divergence in terms of the mean.

Example 3. The variance function of the Gaussian family is $V(\mu) = 1$. Hence, with Φ a standard normal random variable with probability density $\frac{1}{\sqrt{\tau}} \exp\left(-\frac{x^2}{2}\right)$,

$$D(X \| \Phi) \geq \frac{1}{2} E[X]^2$$

if $E[X] \leq 0$.

This inequality actually holds if X is Gaussian with variance 1; using the exponential family based on the Gaussian distribution with x^2 as sufficient statistics we get the inequality

$$D(X \| \Phi) \geq \frac{(Var(X) - 1)^2}{6}$$

if $Var(X) \leq 1$.

The next example is about the exponential distribution.

Example 4. The *Gamma distribution* with shape parameter α and scale parameter β reads

$$\Gamma_{\alpha+1,\theta}(x) = \frac{\left(\frac{x}{\theta}\right)^\alpha}{\alpha! \theta} \exp\left(-\frac{x}{\theta}\right), \quad x \geq 0. \quad (1)$$

The variance function of the Gamma distribution $V(m) = \frac{m^2}{\alpha+1}$ is increasing. Hence

$$D(X \| \Gamma_{\alpha+1,\theta}) \geq \frac{(E[X] - m)^2}{\frac{2m^2}{\alpha+1}}$$

if $E[X] \leq m$. Note that for $\alpha = 0$ we get the *exponential distribution* as a special case.

The next example is about the binomial distribution.

Example 5. The *binomial distribution* has point probabilities

$$\text{bin}(n, p, j) = \binom{n}{j} p^j (1-p)^{n-j}, \quad j = 0, 1, 2, \dots, n.$$

The variance function is $V(m) = m - m^2/n$. The variance function has maximum for $m = n/2$. Hence

$$D(X \| \text{bin}(n, p, j)) \geq \frac{(E[X] - np)^2}{2np(1-p)}$$

if $E[X] \leq np \leq n/2$ or if $E[X] \geq np \geq n/2$. For $p = 1/2$ the inequality

$$D(X \| \text{bin}(n, p, j)) \geq \frac{2(E[X] - \frac{n}{2})^2}{n}$$

that holds for all random variables.

The next example is about the Poisson distribution.

Example 6. The *Poisson distributions* with point probabilities

$$\frac{\lambda^j}{j!} \exp(-\lambda), \quad j = 0, 1, 2, \dots$$

has variance function $V(\lambda) = \lambda$, which is increasing. Hence

$$D(X \| \text{Po}(\lambda)) \geq \frac{(E[X] - \lambda)^2}{2\lambda}$$

for $E[X] \leq \lambda$.

Example 7. The *negative binomial distribution* $NB(r, p)$ with success probability p and number of failures r has point probabilities

$$\binom{k+r-1}{k} (1-p)^r p^k, \quad k = 0, 1, 2, \dots$$

Its variance function $V(m) = \frac{m(m+r)}{r}$ is increasing. Hence

$$D(X \| NB(r, p)) \geq \frac{(E[X] - m)^2}{2\frac{m}{r}(m+r)}.$$

For $r = 1$ we get the geometric distribution as a special case.

The next examples involve cubic variance functions.

Example 8. The *inverse Gaussian distribution* IG has density

$$\left[\frac{\lambda}{\tau x^3}\right]^{1/2} \exp\left(\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right).$$

The variance function $V(\mu) = \mu^3/\lambda$ is increasing so

$$D(X \| IG(\mu, \lambda)) \geq \frac{\lambda(E[X] - \mu)^2}{2\mu^3}$$

if $E[X] \leq \mu$.

Similar results hold for the *generalized Poisson distributions* and for the *inverse binomial distributions* [4]–[7].

III. GENERAL RESULTS FOR GAMMA DISTRIBUTIONS

To simplify the exposition in this section we will assume that the scale parameters θ of the Gamma distributions equal 1.

A. A conjecture for the Gamma case

The Gamma distribution reads

$$\Gamma_{\alpha+1,1}(x) = \frac{x^\alpha}{\alpha!} \exp(-x), \quad x \geq 0.$$

The Laguerre polynomials are given by the Rodrigues formula

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x}), \quad \alpha > -1.$$

The Laguerre polynomials are orthogonal with respect the Gamma distribution, but they are not normalized and they do not all have positive leading coefficient. We thus introduce the normalized Laguerre polynomials by

$$\tilde{L}_n^\alpha(x) = (-1)^n \frac{L_n^\alpha(x)}{\binom{n+\alpha}{n}^{1/2}}.$$

In Example 4 we saw that inequality (1) holds for any random variable X satisfying $\mathbb{E}[\tilde{L}_1^\alpha(X)] < 0$. We conjecture that a similar result holds for the normalized Laguerre polynomials of order 2.

Conjecture 9. *For any random variable X and for any $k \in \mathbb{N}$ we have*

$$D(X \| \Gamma_{\alpha+1,1}) \geq \frac{\mathbb{E}[\tilde{L}_2^\alpha(X)]^2}{2} \quad (2)$$

if $\mathbb{E}[\tilde{L}_2^\alpha(X)] \leq 0$.

Lemma 10. *Let $\{Q_\beta; \beta \in \Gamma\}$ denote an exponential family with x as sufficient statistics so that*

$$\frac{dQ_\beta}{dQ_0} = \frac{\exp(\beta \cdot x)}{Z(\beta)}.$$

If $\mu_0 = 0$ and $V(0) = 1$ and $\mathbb{E}_{Q_0}[X^3] > 0$ then there exists $\varepsilon > 0$ such that

$$D(Q^\mu \| Q_0) \geq \frac{\mu^2}{2}$$

holds for $\mu \in [-\varepsilon, 0]$.

Proof: From Lemma 1 we know that there exists η between μ and μ_0 such that

$$\begin{aligned} D(Q^\mu \| Q_0) &= D(Q^\mu \| Q^{\mu_0}) \\ &= \frac{(\mu - \mu_0)^2}{2} \cdot \frac{1}{V(\eta)}. \end{aligned}$$

Therefore it is sufficient to prove there exists $\varepsilon > 0$ such that $V(\eta) \leq 1$ for $[-\varepsilon, 0]$. This follows from the fact that

$$\begin{aligned} \frac{dV(\eta)}{d\eta} &= \frac{\frac{dV(\eta)}{d\beta}}{\frac{d\eta}{d\beta}} \\ &= \frac{\frac{d^3}{d\beta^3} \ln Z(\beta)}{\frac{d^2}{d\beta^2} \ln Z(\beta)} \\ &= \frac{\mathbb{E}[(X - \eta)^3]}{\text{Var}(X)} \end{aligned}$$

where the mean and variance are taken with respect to the element in the exponential family with mean η . Since $\frac{\mathbb{E}[X^3]}{\text{Var}(X)} > 0$ for $\beta = 0$ we have that $\frac{\mathbb{E}[(X - \eta)^3]}{\text{Var}(X)} > 0$ for β in a neighborhood of 0 so $V(\eta)$ is an increasing of η and the result then follows from $V(0) = 1$. ■

We can now formulate the following result.

Proposition 11. *For all $n \in \mathbb{N}_0$ and all $\alpha > -1$ we have*

$$\int_0^\infty \left(\tilde{L}_n^\alpha(x)\right)^3 \Gamma_{\alpha+1,1}(x) dx > 0.$$

Proof: We have

$$\begin{aligned} \int_0^\infty \left(\tilde{L}_n^\alpha(x)\right)^3 \Gamma_\alpha(x) dx &= \int_0^\infty (-1)^{3n} \frac{(L_n^\alpha(x))^3}{\binom{n+\alpha}{n}^{3/2}} \Gamma_{\alpha+1,1}(x) dx \\ &= (-1)^n \binom{n+\alpha}{n}^{-3/2} \int_0^\infty (L_n^\alpha(x))^3 \Gamma_{\alpha+1,1}(x) dx, \end{aligned}$$

which, according to [8, p. 57], is strictly positive. ■

Theorem 12. *For any $n \in \mathbb{N}_0$ and any $\alpha > -1$ there exists $\epsilon > 0$ that may depend on α and n such that*

$$D(X \| \Gamma_{\alpha+1,1}(X)) \geq \frac{1}{2} \left(\mathbb{E}[\tilde{L}_n^\alpha(x)] \right)^2$$

for any random variable X satisfying $\mathbb{E}[\tilde{L}_n^\alpha(x)] \in [-\epsilon, 0]$.

In the Gaussian case, we have the similar

Corollary 13. *For any $n \in \mathbb{N}_0$ there exists $\epsilon > 0$ such that*

$$D(X \| \Phi) \geq \frac{1}{2} (\mathbb{E}[H_{2n}(X)])^2$$

for any random variable X satisfying $H_{2n}(X) \in [-\epsilon, 0]$.

This inequality has previously been proved by considering the Hermite polynomials as limits of Poisson-Charlier polynomials for which a similar inequality holds [9].

IV. LAGUERRE POLYNOMIALS OF DEGREE 2

We shall use the following lemma.

Lemma 14. *Assume that*

$$\int_0^\infty \left(\tilde{L}_k^\alpha(X)\right)^2 \exp\left(\beta_0 \tilde{L}_k^\alpha(X)\right) \frac{x^\alpha}{\alpha!} \exp(-x) dx \leq 1 \quad (3)$$

then the conjecture holds for all $\mathbb{E}[\tilde{L}_k^\alpha(X)] \in [\beta_0, 0]$.

Proof: Let Q_β denote the distribution with density

$$\frac{dQ_\beta}{d\Gamma_{\alpha+1,1}} = \frac{\exp\left(\beta \cdot \tilde{L}_k^\alpha(x)\right)}{Z(\beta)}.$$

We have to prove that

$$D(Q_\beta \| \Gamma_{\alpha+1,1}) \geq \frac{1}{2} (\mu_\beta)^2.$$

for $\mu_\beta \in [\beta_0, 0]$. We have $D(Q_\beta \| \Gamma_{\alpha+1,1}) = \beta \mu_\beta - \ln(Z(\beta))$ and $\mu_\beta = \frac{Z'(\beta)}{Z(\beta)}$. The inequality is satisfied for $\beta = 0$ so we differentiate with respect to β and have to prove that

$$\mu_\beta + \beta \frac{d\mu_\beta}{d\beta} - \frac{Z'(\beta)}{Z(\beta)} \leq \frac{1}{2} \cdot 2\mu_\beta \frac{d\mu_\beta}{d\beta}$$

which is equivalent to

$$\beta \leq \mu_\beta.$$

Since we have assumed that $\mu_\beta \in [\beta_0, 0]$ it is sufficient to prove the inequality for $\beta \in [\beta_0, 0]$. The inequality is satisfied for $\beta = 0$ so we differentiate once more so that we have to prove the inequality

$$1 \geq d\mu_\beta = \frac{Z''(\beta) Z(\beta) - (Z'(\beta))^2}{(Z(\beta))^2}.$$

Hence it is sufficient to prove that

$$\frac{Z''(\beta)}{Z(\beta)} \leq 1$$

which is equivalent to

$$Z''(\beta) \leq Z(\beta).$$

Since $Z(\beta) \geq 1$ for all β so it is sufficient to prove that $Z''(\beta) \leq 1$ for $\beta \in [\beta_0, 0]$. The function $\beta \rightarrow Z''(\beta)$ is convex and

$$Z''(0) = \int_0^\infty \left(\tilde{L}_k^\alpha(X) \right)^2 \frac{x^\alpha}{\alpha!} \exp(-x) dx = 1.$$

Therefore it is sufficient to check that $Z''(\beta_0) \leq 1$, which is exactly what is stated in (3). ■

A. Large values of the shape parameter

If the scale parameter is fixed at 1 and the shape parameter tends to infinity then the Gamma distribution will tend to a Gaussian. We know that one can get a lower bound on the information divergence in terms of the Hermite polynomial of order 2 so we should expect this also to hold for large values of the shape parameter. This is indeed the case as stated in the following theorem.

Theorem 15. For any $\alpha \geq 6^{1/2}$

$$D(X \| \Gamma_{\alpha+1,1}(X)) \geq \frac{1}{2} \left(E \left[\tilde{L}_2^\alpha(X) \right] \right)^2$$

for any random variable X satisfying $E \left[\tilde{L}_n^\alpha(X) \right] \leq 0$.

Proof: Let β_0 denote the negative solution to the equation $\beta^2 \exp(\beta^2) = 1$. The value is approximately $\beta_0 = -0.75309$. The function $f(x) = x^2 \exp(\beta_0 x)$ is decreasing for $x \in]-\infty, 0]$, increasing for $x \in [0, -2/\beta_0]$ and decreasing for $x \in [-2/\beta_0, \infty[$. The local maximum in $x = -2/\beta_0$ has the value $0.9545 < 1$. According to the definition of β_0 we have $f(\beta_0) = 1$ so $f(x) \leq 1$ for $x \geq \beta_0$. The second normalized Laguerre polynomial is

$$\begin{aligned} \tilde{L}_n^\alpha(x) &= \frac{L_n^\alpha(x)}{\left(\frac{(2+\alpha)(1+\alpha)}{2} \right)^{1/2}} \\ &= \frac{x^2 - 2(\alpha+2)x + (\alpha+2)(\alpha+1)}{(2(2+\alpha)(1+\alpha))^{1/2}}. \end{aligned}$$

The minimum is attained for $x = 2(\alpha+1)$ and has the value

$$-2^{-1/2} \left(1 + \frac{1}{\alpha+1} \right)^{1/2}.$$

This is an increasing function of α that tends to $-2^{-1/2} > \beta_0$ for x tending to ∞ . We solve the equation

$$-2^{-1/2} \left(1 + \frac{1}{\alpha+1} \right)^{1/2} = \beta_0$$

and get

$$\alpha_0 = \frac{1}{2\beta_0^2 - 1} - 1 = 6.4466.$$

Therefore $\tilde{L}_n^\alpha(x) \geq \beta_0$ for all x if $\alpha \geq \alpha_0$. Hence $f(\tilde{L}_n^\alpha(x)) \leq 1$ for all x if $\alpha \geq \alpha_0$. In particular

$$\begin{aligned} \int_0^\infty \left(\tilde{L}_2^\alpha(x) \right)^2 \exp(\beta_0 \tilde{L}_k^\alpha(x)) \frac{x^\alpha}{\alpha!} \exp(-x) dx \\ = \int_0^\infty f(\tilde{L}_2^\alpha(x)) \frac{x^\alpha}{\alpha!} \exp(-x) dx \leq 1. \end{aligned}$$

This proves that the inequality holds whenever $E \left[\tilde{L}_2^\alpha(X) \right] \in [\beta_0, 0]$. The condition $E \left[\tilde{L}_2^\alpha(X) \right] \geq \beta_0$ is automatically fulfilled if $\alpha \geq \alpha_0$ and the theorem follows. ■

B. Chi square distributions

The χ^2 -distributions are Gamma distributions with half integral value of α and scale parameter $1/2$. We will check our conjecture for $\alpha < 6^{1/2}$ and half integral values. For notational convenience we will assume that the shape parameter is 1 and note that results for χ^2 -distributions are obtained by a simple scaling. According to Lemma 14 it is sufficient to calculate the integral (3) when

$$\beta_0 = \min_x \tilde{L}_2^\alpha(X) = -2^{-1/2} \left(1 + \frac{1}{\alpha+1} \right)^{1/2}.$$

The results are given in the following.

α	β_0	\int
$-1/2$	-1.225	0.95407
0	-1	0.63113
$1/2$	-0.9129	0.55406
1	-0.8660	0.52046
$11/2$	-0.8367	0.5018
2	-0.8165	0.48997
$21/2$	-0.8018	0.48181
3	-0.7906	0.47584
$31/2$	-0.7817	0.47128
4	-0.7746	0.46769
$41/2$	-0.7687	0.46478
5	-0.7638	0.46238
$51/2$	-0.7596	0.46037
6	-0.7559	0.45865

As we see all values of the integral are less than 1 so the conjecture holds for all half integral values of α . This gives us the following theorem.

Theorem 16. Assume that $\alpha > -1$ and that 2α is an integer. Then, for any random variable satisfying $E \left[\tilde{L}_2^\alpha(X) \right] \leq 0$, we have

$$D(X \| \Gamma_{\alpha+1,1}) \geq \frac{1}{2} \left(E \left[\tilde{L}_2^\alpha(X) \right] \right)^2.$$

Example 17. For $\alpha = 0$ we get the exponential distribution with density

$$\exp(-x), x > 0.$$

The Laguerre polynomial of order two is $L_2(x) = \frac{1}{2}x^2 - 2x + 1$. We will rewrite our inequality in terms of mean and variance. For any random variable satisfying $\text{Var}(X) \leq 1$ and $E[X] = 1$ we get the inequality

$$D(X \| \text{Exp}(1)) \geq \frac{1}{8} (\text{Var}(X) - 1)^2.$$

The χ^2 -distribution with 1 degree of freedom corresponds to a Gamma distribution with shape parameter $\alpha + 1 = 1/2$ and scale parameter 2. It has density

$$\frac{x^{-1/2}}{\tau^{1/2}} \exp\left(-\frac{x}{2}\right)$$

This distribution is important because it is the distribution of the square of a standard Gaussian random variable. Hence, results for the χ^2 distribution translate into results for Hermite moments. In order to follow the notation from the previous section we first prove results for the Gamma distribution with shape parameter $\alpha + 1 = 1/2$ and scale parameter 1 and then translate the results.

We have

$$\tilde{L}_2^{-1/2}(x) = \frac{x^2}{2} - \frac{3}{2}x + \frac{3}{8}$$

and the normalized version

$$\tilde{L}_2^{-1/2}(x) = \frac{x^2 - 3x + \frac{3}{4}}{(3/2)^{1/2}}.$$

This gives us the following theorem.

Theorem 18. *For any random variable satisfying $\mathbb{E}[\tilde{L}_2^{-1/2}(X)] \leq 0$, we have*

$$D(X\|\Gamma_{1/2,1}) \geq \frac{1}{2} \left(\mathbb{E}[\tilde{L}_2^{-1/2}(X)] \right)^2.$$

Corollary 19. *For a random variable X satisfying $\mathbb{E}[X] = 1$ and $\text{Var}(X) \leq 2$ we have*

$$D(X\|\chi_1^2) \geq \frac{(\text{Var}(X) - 2)^2}{48}.$$

Proof: The result follows from the following computation.

$$\begin{aligned} D(X\|\chi_1^2) &= D\left(\frac{X}{2}\|\Gamma_{\alpha+1,1}\right) \\ &\geq \frac{1}{2} \left(\mathbb{E}\left[\tilde{L}_2^{-1/2}\left(\frac{X}{2}\right)\right] \right)^2 \\ &= \frac{1}{2} \left(\mathbb{E}\left[\frac{\left(\frac{X}{2}\right)^2 - 3\left(\frac{X}{2}\right) + \frac{3}{4}}{\left(\frac{3}{2}\right)^{1/2}}\right] \right)^2 \\ &= \frac{1}{48} \left(\text{Var}(X) + \mathbb{E}[X]^2 - 6\mathbb{E}[X] + 3 \right)^2 \\ &= \frac{(\text{Var}(X) - 2)^2}{48}. \end{aligned}$$

These inequalities can be translated into inequalities for Hermite polynomials. ■

Corollary 20. *For any random variable satisfying $\mathbb{E}[H_4(X)] \leq 0$ we have*

$$D(X\|\Phi) \geq \frac{1}{2} (\mathbb{E}[H_4(X)])^2.$$

If $\text{Var}(X) = 1$ this is equivalent to

$$D(X\|\Phi) \geq \frac{\kappa^2}{48} \quad (4)$$

if X is platykurtic and κ denotes the excess kurtosis.

The inequality (4) was proved in [2] with a different technique.

V. COUNTEREXAMPLE

With all these positive results in mind one may conjecture that

$$D(X\|\Gamma_{\alpha+1,1}) \geq \frac{\mathbb{E}[\tilde{L}_k^\alpha(X)]^2}{2} \quad (5)$$

would hold for all k as long as $\mathbb{E}[\tilde{L}_k^\alpha(X)] \leq 0$, but this is not the case. Here we will describe a counterexample for $k = 3$ and $\alpha = -1/2$. We will fix $\mathbb{E}[\tilde{L}_3^\alpha(X)] = -3$. In this case the information projection of $\Gamma_{1/2,1}$ onto the set of distributions satisfying $\mathbb{E}[\tilde{L}_3^\alpha(X)] = -3$ equals the distribution Q_β with density

$$\frac{dQ_\beta}{d\Gamma_{1/2,1}}(x) = \frac{\exp(\beta \tilde{L}_3^{1/2}(x))}{\int_0^\infty \exp(\beta \tilde{L}_3^{1/2}(x) \Gamma_{\alpha+1,\theta}(x) dx)}.$$

Numerical calculations gives $\beta = -1.83125$ and $D(Q_\beta\|\Gamma_{1/2,1}) = 3.3195$, which is not greater than $\frac{1}{2}(-3)^2$. The counterexample implies that there exists a random variable X such that

$$D(X\|\Phi) \not\geq \frac{1}{2} (\mathbb{E}[H_6(X)])^2.$$

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